J. Fluid Mech. (1966), *vol.* 42, *part* 4, *pp.* 705–720
Printed in Great Britain

Magnetohydrodynamic drag on an oscillating sphere

By ROBIN O. MOTZ
Plasma Laboratory, Columbia University, New York City†

(Received 31 August 1965)

The case of a dielectric sphere executing small-amplitude oscillations in an incompressible, conducting fluid with a uniform magnetic field aligned along the axis of oscillation is studied. For the case of large Hartmann and Reynolds numbers, the MHD equations are linearized. Solutions correct to $O(R_m)$ are obtained for the velocity, pressure, current, and electric and magnetic fields in the fluid. The MHD drag on an oscillating dielectric sphere in mercury was experimentally determined in aligned fields to 0·8 Wb/m^2. The experimental results agree well with the theory. The theory and experiment were extended to include the régime where the perturbed velocity is not small compared with the velocity present in the absence of a magnetic field. The validity of using the inviscid velocity to compute the current is verified by measuring the angular dependence of the induced electric field.

1. Introduction

We consider here a dielectric sphere executing rapid, small-amplitude oscillations in an electrically conducting fluid, with a uniform magnetic field aligned along the axis of oscillation.

The problem in the absence of a magnetic field has been studied by Green (1833) and Stokes (1843) for an inviscid fluid, and by Stokes (1850) for a viscous fluid. Stokes's viscous solution, which involves an expansion of the velocity in the dimensionless amplitude so that the convective term may be neglected in the substantive derivative, has been verified qualitatively for the case of a sphere by Carstens (1952) and quantitatively for a cylinder by Stuart & Woodgate (1955). Stokes's work shows that the inviscid velocity field may be used to describe the bulk motion of the fluid because oscillation at high Reynolds numbers confines the effect of viscosity to a thin boundary layer surrounding the body. Since we are dealing with large Hartmann numbers, outside this boundary layer any vorticity present will be due to electromagnetic rather than viscous forces. Furthermore, outside this boundary layer, Stokes's solution reduces rapidly to the inviscid solution.

Within the past ten years, several authors have studied the effect of a magnetic field on the uniform motion of a sphere through a conducting fluid. The inviscid case has been studied by Reitz & Foldy (1961), who considered a uniform, external field, and by Ludford & Murray (1960), who considered a magnetized sphere.

† Present address: Department of Physics, Stevens Institute of Technology, Hoboken, New Jersey.

Reitz & Foldy showed that, correct to first order in the magnetic Reynolds number, the drag can be computed by equating the work done to keep the sphere moving with constant velocity to the Joule loss in the fluid. They computed their current from the applied magnetic field, and the known inviscid fluid velocity which would be present in the absence of a magnetic field.

The case of an oscillating sphere has very recently been considered by Singh (1965). In the limit of zero viscosity and compressibility, his drag result agrees with Reitz & Foldy. However, Singh restricted himself to weak magnetic fields (less than 50 G), so that the magnetic pressure is at most comparable to the dynamic pressure; we do not make this restriction.

There have been two experiments dealing with the MHD sphere-drag problem. Dorman & Mikhailov (1963) measured the electric and magnetic fields generated by the uniform motion of a sphere in mercury with an aligned magnetic field for small Stuart number. They found functional (qualitative) agreement with the fields predicted by Reitz & Foldy. The only known experiment that attempted to measure the drag directly was that of Maxworthy (1962). He dropped spheres in liquid sodium with an aligned magnetic field, and computed the drag from the terminal velocity. This was a check of the linearized MHD equations with viscosity present. Maxworthy's experiment showed only qualitative agreement with the work of Chester (1952) and with a prediction of Childress (1961) in that the drag was observed to scale linearly with (M/R_e) at low magnetic fields.

2. Equations for an oscillating body

The MHD equations for an incompressible, inviscid fluid are:

$$\left.\begin{aligned} \rho\{(\partial \mathbf{V}/\partial t)+(\mathbf{V}.\nabla)\mathbf{V}\} &= -\nabla p+\mathbf{J}\times\mathbf{B}, \quad \nabla.\mathbf{V}=0,\\ \nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t, \quad \nabla\times\mathbf{B} &= \mu\mathbf{J}, \quad \nabla.\mathbf{B}=0,\\ \mathbf{J} = \sigma(\mathbf{E}+\mathbf{V}\times\mathbf{B}), &\quad \nabla.\mathbf{J}=0. \end{aligned}\right\} \tag{1}$$

Consider now an axisymmetric body of length a in the fluid, oscillating with amplitude z_0 and angular frequency ω, so that its velocity is

$$\mathbf{U}(t) = U_0 e^{i\omega t}\hat{\mathbf{z}}, \quad U_0 = z_0\omega, \tag{2}$$

where $\hat{\mathbf{z}}$ is the unit vector in the z-direction, and apply an external magnetic field $\mathbf{H}_0 = H_0\hat{\mathbf{z}}$. Equations (1) are made dimensionless by substitution of the following new variables

$$\left.\begin{aligned} V^* = V/U_0, \quad E^* = E/\mu U_0 H_0, \quad t^* = \omega t, \quad r^* = r/a,\\ H^* = H/H_0, \quad J^* = J/(H_0/a), \quad p^* = p/\rho V_A^2, \end{aligned}\right\} \tag{3}$$

where $V_A = \sqrt{(\mu H_0^2/\rho)}$ is the Alfvén velocity. Then

$$\nabla^*.\mathbf{V}^* = 0, \quad \nabla^*.\mathbf{H}^* = 0, \quad \nabla^*.\mathbf{J}^* = 0, \tag{4a, b, c}$$

$$\frac{1}{\alpha}\left\{\frac{\partial \mathbf{V}^*}{\partial t^*}+\alpha(\mathbf{V}^*.\nabla^*)\mathbf{V}^*\right\} = -\beta\nabla^* p^*+\beta\mathbf{J}^*\times\mathbf{H}^*, \tag{5}$$

$$\nabla^*\times\mathbf{H}^* = \mathbf{J}^*, \quad \alpha\nabla^*\times\mathbf{E}^* = -\partial\mathbf{H}^*/\partial t^*, \tag{6a, b}$$

$$\mathbf{J}^* = R_m(\mathbf{E}^*+\mathbf{V}^*\times\mathbf{H}^*), \tag{7}$$

and in (5)–(7) the following dimensionless parameters are used:

$$R_m = U_0 a\mu\sigma, \quad \beta = V_A^2/U_0^2 = (\mu H_0^2)/(\rho U_0^2), \quad \alpha = U_0/(a\omega) = z_0/a. \tag{8}$$

Following Stokes, $\mathbf{V}^*$ is expanded as a power series in α. Keeping only the first order terms, (5) becomes

$$\partial \mathbf{V}^*/\partial t^* = -\beta \nabla^* p^* + \beta \mathbf{J}^* \times \mathbf{H}^*. \tag{9}$$

That is, for small-amplitude oscillations, the substantive derivative may be replaced by the partial time derivative, correct to $O(\alpha)$.

The interdependence of the velocity and the magnetic field is simplified by expanding all quantities of interest as a power series in the magnetic Reynolds number R_m:

$$\mathbf{G}^* = \mathbf{g}_0 + R_m \mathbf{g}_1 + O(R_m^2).$$

Upon expanding (4), (6), (7), (9) and collecting coefficients of like powers of R_m,

$$\nabla . \mathbf{v}_1 = 0, \quad \nabla . \mathbf{h}_1 = 0, \quad \nabla . \mathbf{j}_1 = 0, \tag{10a,b,c}$$

$$\nabla \times \mathbf{h}_1 = \mathbf{j}_1, \quad \alpha \nabla \times \mathbf{e}_1 = -\partial \mathbf{h}_1/\partial t, \tag{11a,b}$$

$$\mathbf{j}_0 = 0, \quad \mathbf{j}_1 = (\mathbf{e}_0 + \mathbf{v}_0 \times \mathbf{h}_0), \tag{12a,b}$$

$$\partial \mathbf{v}_1/\partial t = -\beta \nabla p_1 + \beta \mathbf{j}_1 \times \mathbf{h}_0. \tag{13}$$

(10)–(13) are a complete set of equations which, when boundary conditions are applied, can be solved for all quantities of first order in R_m, once $\mathbf{v}_0$ and $\mathbf{h}_0$ are known.

3. Solution for an axisymmetric body of arbitrary shape

Since $\mathbf{j}_0 = 0$, $\mathbf{h}_0$ is irrotational. Since it is also solenoidal, it may be chosen as uniform and constant, so we set $\mathbf{h}_0 = \hat{\mathbf{z}}$. As $\mathbf{h}_0$ is chosen to be time-independent, $\mathbf{e}_0$ is also irrotational. From (12b) and (10c), $\mathbf{e}_0$ is solenoidal for a body oscillating along an axis of symmetry since then $\mathbf{v}_0$ has no azimuthal component. We therefore choose $\mathbf{e}_0 = 0$. (12b) then gives

$$\mathbf{j}_1 = \mathbf{v}_0 \times \hat{\mathbf{z}}. \tag{14}$$

(14) may be solved immediately for $\mathbf{j}_1$ once $\mathbf{v}_0$ is known. We seek, however, an expression for $\mathbf{j}_1$ which does not contain $\mathbf{v}_0$ explicitly. This is because it is desirable to express $\mathbf{v}_1$, $\mathbf{h}_1$ and p_1 in such a manner that, once $\mathbf{j}_1$ is known, they may be written down with minimal calculation. For zeroth order in R_m there is no dissipation. Applying Kelvin's theorem

$$\nabla \times \mathbf{v}_0 = \boldsymbol{\omega}_0 = 0. \tag{15}$$

(Note that for oscillatory motion (15) applies only to $O(\alpha)$.) Taking the curl of (14) twice yields

$$\nabla \times \mathbf{j}_1 = (\mathbf{h}_0 . \nabla)\, \mathbf{v}_0 - (\mathbf{v}_0 . \nabla)\, \mathbf{h}_0 = (\mathbf{h}_0 . \nabla)\, \mathbf{v}_0,$$

$$\nabla \times (\nabla \times \mathbf{j}_1) = -\nabla^2 \mathbf{j}_1 + \nabla(\nabla . \mathbf{j}_1) = -\nabla^2(\mathbf{j}_1) = (\mathbf{h}_0 . \nabla)\, \boldsymbol{\omega}_0 = 0,$$

so that any current generated by an irrotational velocity satisfies

$$\nabla^2 \mathbf{j}_1 = 0. \tag{16}$$

Since $\mathbf{j}_1$ must be azimuthal as well as solenoidal, then if it vanishes at infinity it must have the form (Morse & Feshbach 1953)

$$\mathbf{j}_1 = \sum_n A_n \frac{P'_n(\cos\theta)}{r^{n+1}} e^{it} \hat{\boldsymbol{\phi}}, \tag{17}$$

where the A_n are obtained from (14), $P'_n(\cos\theta)$ is the first associated Legendre polynomial, and $\hat{\boldsymbol{\phi}}$ is the unit azimuthal vector.

$\mathbf{h}_1$ is found by substitution of (17) in (11*a*)

$$\nabla \times \mathbf{h}_1 = \mathbf{j}_1 = \sum_n \frac{A_n P'_n}{r^{n+1}} e^{it} \hat{\boldsymbol{\phi}}. \tag{18}$$

The most general function that satisfies (18), is solenoidal, and vanishes at infinity is

$$\left.\begin{aligned} \mathbf{h}_1 = & -\sum_n A_n e^{it} \left\{ \frac{n(n-7)}{2(2n+1)} \frac{P_n}{r^n} \hat{\mathbf{r}} + \frac{(n-2)}{2(2n+1)} \frac{P'_n}{r^n} \hat{\boldsymbol{\theta}} \right\}, \\ & - \sum_k B_k e^{it} \left\{ \frac{(k+1)P_k}{r^{k+2}} \hat{\mathbf{r}} + \frac{P'_k}{r^{k+2}} \hat{\boldsymbol{\theta}} \right\}, \end{aligned}\right\} \tag{19}$$

where the B_k's are to be determined by boundary matching h_1 at the surface of the oscillating body; $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors.

To solve for $\mathbf{v}_1$, the vorticity due to electromagnetic forces is needed. Taking the curl of (13) and substituting $\mathbf{h}_0 = \hat{\mathbf{z}}$,

$$\frac{\partial \boldsymbol{\omega}_1}{\partial t} = \beta \frac{\partial \mathbf{j}_1}{\partial z}, \tag{20}$$

and, substituting for $\mathbf{j}_1$ from (17),

$$\begin{aligned} \boldsymbol{\omega}_1 &= -i \sum_n \beta A_n \left\{ (n+1)\cos\theta \frac{\partial P_n}{\partial \theta} + \sin\theta \frac{\partial^2 P_n}{\partial \theta^2} \right\} \frac{e^{it}}{r^{n+2}} \hat{\boldsymbol{\phi}} \\ &= i \sum_n \beta n A_n \frac{P'_{n+1}}{r^{n+2}} e^{it} \hat{\boldsymbol{\phi}}. \end{aligned} \tag{21}$$

$\mathbf{v}_1$ must satisfy $\nabla \times \mathbf{v}_1 = \boldsymbol{\omega}_1$, (10*a*), and the boundary conditions

$$\left.\begin{aligned} v_1 \to 0 \quad & \text{as} \quad r \to \infty, \\ \mathbf{v}_1 . \hat{\mathbf{n}} = 0 \quad & \text{on the body.} \end{aligned}\right\} \tag{22}$$

The second condition of (22) follows from the inviscid boundary condition $\mathbf{V}.\hat{\mathbf{n}} = \mathbf{U}.\hat{\mathbf{n}}$ on the body, which is already satisfied by $\mathbf{v}_0$. Solving for $\mathbf{v}_1$, we find

$$\begin{aligned} \mathbf{v}_1 = i\beta \sum_n \frac{A_n e^{it}}{r^{n+1}} & \left[\left\{ \sin\theta P'_n - \frac{n^2(n+1)}{2(2n+1)} P_{n+1} \right\} \hat{\mathbf{r}} + \left\{ \cos\theta P'_n - \frac{n(n+1)}{2(2n+1)} P'_{n+1} \right\} \hat{\boldsymbol{\theta}} \right] \\ & - \sum_m C_m e^{it} \left\{ \frac{(m+1)}{r^{m+2}} P_m \hat{\mathbf{r}} + \frac{P'_m}{r^{m+2}} \hat{\boldsymbol{\theta}} \right\}, \end{aligned} \tag{23}$$

where the C_m are chosen to satisfy (22) for the body.

The pressure drag is found by integrating the perturbed pressure p_1 over the surface of the body. This pressure may be determined by direct integration of equation (13), which may be written

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\beta \nabla p_1 + \beta \mathbf{j}_1 \times \hat{\mathbf{z}},$$

upon substitution for $\mathbf{j}_1$ and $\mathbf{v}_1$. Note that the resultant drag will just depend on the current in the fluid; nowhere in (13) does the conductivity of the body enter explicitly. That is, (13) does not directly take into account the current and magnetic field inside the body. This means that the internal field contributes a term to the pressure drag of $O(R_m^2)$ at best. One might think that the electromagnetic effect of the body on the pressure drag would appear if (13) were rewritten so that $\mathbf{h}_1$ appeared explicitly, since the B_k's depend on the conductivity of the body through boundary conditions. It will be shown that, although the pressure can be written as a function of $\mathbf{v}_1$ and $\mathbf{h}_1$, the conductivity of the body will not appear in the pressure drag term of $O(R_m)$.

(13) is rewritten to simplify the computation of p_1 by using the vector identity $(\nabla \times \mathbf{h}) \times \mathbf{h} = (\mathbf{h}.\nabla)\,\mathbf{h} - \frac{1}{2}\nabla h^2$, and keeping only $O(R_m)$

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\beta \nabla (p_1 + \mathbf{h}_1 . \hat{\mathbf{z}}) + \beta \frac{\partial \mathbf{h}_1}{\partial z}. \tag{24a}$$

The divergence of (24a) gives

$$\nabla^2 (p_1 + \mathbf{h}_1 . \hat{\mathbf{z}}) = 0,$$

so that

$$p_1 + \mathbf{h}_1 . \hat{\mathbf{z}} = \sum_{\gamma} \frac{D_\gamma P_\gamma}{r^{\gamma+1}}. \tag{24b}$$

The pressure is found by substituting $\mathbf{v}_1$ and $\mathbf{h}_1$, once they are known, into (24a), and using (24b) to determine γ by matching powers of r. Then p_1 is found by evaluating D_γ through the substitution of (24b) into the z-component of (24a)

$$\frac{\partial}{\partial t}(\mathbf{v}_1 . \hat{\mathbf{z}}) = -\beta \frac{\partial p_1}{\partial z}. \tag{24c}$$

4. Solution for a sphere

The potential velocity for a sphere of radius a oscillating with small amplitude is (Lamb 1952)

$$\mathbf{V}_0 = (\tfrac{1}{2} U_0 a^3 / r^3)\,(2 \cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}})\, e^{i\omega t}, \quad \mathbf{v}_0 = \tfrac{1}{2} r^{-3} (2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}})\, e^{it}, \tag{25}$$

so that

$$\mathbf{j}_1 = -\tfrac{3}{2} r^{-3} \sin\theta \cos\theta \, e^{it} \hat{\boldsymbol{\phi}}. \tag{26}$$

By comparison with (17), $n = 2$, $A_n = -\frac{1}{2}$. The vorticity is then

$$\boldsymbol{\omega}_1 = (i\beta A_2 / r^4)\,(12 \sin\theta - 15 \sin^3\theta)\, e^{it} \hat{\boldsymbol{\phi}} \tag{27}$$

and from (23)

$$\mathbf{v}_1 = -\frac{i\beta A_2}{r^3} \{(\sin\theta P_2' - \tfrac{6}{5} P_3)\,\hat{\mathbf{r}} + (\cos\theta P_2' - \tfrac{3}{5} P_3')\,\hat{\boldsymbol{\theta}}\} e^{it} + \nabla \sum_m \frac{C_m P_m}{r^{m+1}} e^{it},$$

$$= -\frac{i\beta A_2}{r^3} \{6 \cos\theta \sin^2\theta \hat{\mathbf{r}} + \tfrac{3}{2} \sin^3\theta \hat{\boldsymbol{\theta}}\} e^{it} + \nabla \sum_m \frac{C_m P_m}{r^{m+1}} e^{it}. \tag{28}$$

For a sphere, (22) becomes $\mathbf{v}_1 . \hat{\mathbf{r}} = 0$. Since

$$6 \sin^2 \theta \cos \theta = -\tfrac{12}{5}(P_1 + P_3),$$

then to satisfy the boundary condition $m = 1$, or 3. After solving for C_1, C_3

$$\mathbf{v}_1 = -\frac{i\beta A_2}{r^3}\left[\left\{6\cos^3\theta\left(1-\frac{1}{r^2}\right) - \tfrac{18}{5}\cos\theta\left(1-\frac{1}{r^2}\right)\right\}\hat{\mathbf{r}} - \left\{\tfrac{3}{2}\sin^3\theta\left(1+\frac{3}{r^2}\right) - \tfrac{2}{5}\sin\theta\left(2-\frac{9}{r^2}\right)\right\}\hat{\boldsymbol{\theta}}\right]e^{it}. \tag{29}$$

From (19), the magnetic field is

$$\mathbf{h}_1 = \left(\frac{A_2 P_2}{r^2}\hat{\mathbf{r}} + \nabla \sum_k \frac{B_k P_k}{r^{k+1}}\right) e^{it}. \tag{30}$$

k is evaluated by using (29) and (24). Substitution of (30), (29) and (26) in (28) shows that, to match powers of r and $\cos\theta$, $k = 2$ or 0. $k = 0$ must be rejected because it gives rise to an electric field that becomes infinite at $\theta = 0$. Thus

$$\mathbf{h}_1 = \left\{\left(\frac{A_1 P_2}{r^2} - \frac{3B_2 P_2}{r^4}\right)\hat{\mathbf{r}} - \frac{B_2 P_2'}{r^4}\hat{\boldsymbol{\theta}}\right\} e^{it}. \tag{31}$$

From (31) and (11b)

$$\mathbf{e}_1 = e(r) \sin\theta \cos\theta \, e^{it} \hat{\boldsymbol{\phi}}, \tag{32}$$

and $\mathbf{e}_1$ vanishes at $\theta = 0°, 90°$ where $\mathbf{v}_0$ is parallel to $\mathbf{h}_0$. Making the substitution

$$B_2 = \Gamma A_2, \tag{33}$$

(31) and (11b) are used to solve for $\mathbf{e}_1$. From (32), $\mathbf{e}_1$ is seen to be solenoidal. Since it has only one component, the integration of (11b) after substitution of (31) is straightforward

$$\mathbf{e}_1 = \frac{iA_2}{\alpha}\left(\frac{\Gamma}{2r^3} - \frac{1}{6r}\right) e^{it} \hat{\boldsymbol{\phi}}. \tag{34}$$

To compute the pressure, (30) is substituted in (24b) to give

$$p_1 = \sum_\gamma \frac{D_\gamma P_\gamma}{r^{\gamma+1}} - A_2\left\{\cos^3\theta\left(\frac{1}{2r^2} - \frac{15\Gamma}{2r^4}\right) + \cos\theta\left(-\frac{1}{2r^2} + \frac{9\Gamma}{2r^4}\right)\right\} e^{it}. \tag{35}$$

Substitution of (35) and $\mathbf{v}_1$ in (24c) shows that $\gamma = 1$ or 3. Solving for D_γ,

$$D_1 = \tfrac{1}{5}A_2, \quad D_3 = \tfrac{1}{10}A_2(3\Gamma - 6),$$

and the perturbed pressure is

$$p_1 = -A_2\left\{\frac{1}{r^2}\left(\tfrac{3}{5}P_1 + \tfrac{6}{5}P_3\right) + \frac{1}{r^4}\left(\frac{3}{5} - \frac{33\Gamma}{10}\right) P_3\right\} e^{it}. \tag{36}$$

Γ must be evaluated to complete the solution, which involves computing the fields inside the sphere. This has been done (Motz 1964). There are three cases. Let the sphere have unit permeability. Then for an infinite-conductivity sphere

$$\Gamma = \tfrac{1}{3}, \tag{37a}$$

for a dielectric sphere, $$\Gamma = \tfrac{1}{5} \text{ (if } U_0^2 \ll \alpha^2 c^2), \tag{37b}$$

for conduction current larger than displacement current,

$$\Gamma = \lambda/(1+3\lambda), \tag{37c}$$

where $$\lambda = \tfrac{1}{6}\mathscr{R}_e\left\{\frac{\gamma I'_{\frac{3}{2}}(\gamma\sqrt{i})}{I_{\frac{3}{2}}(\gamma\sqrt{i})}+\tfrac{1}{2}\right\}, \quad \gamma = \sqrt{(a^2\omega\mu\sigma_s)}. \tag{38}$$

Here $\mathscr{R}_e$ denotes the real part, and $I_{\frac{3}{2}}$ is the modified Bessel function of half-integral powers of i.

5. Drag and energy dissipation for the oscillating sphere

To compute the drag, the force exerted on the sphere is calculated. We are interested in the instantaneous (i.e. time-dependent) drag because this is the force that appears in the equation of motion of damped oscillation; therefore a time average is not computed. There are two sources of drag: the mechanical (fluid) pressure, and the Maxwell stress.

The pressure drag is computed by integrating the pressure over the surface of the sphere. Since the pressure acts normally, the dimensionless drag is given by

$$D_p = F_p/(\rho V_A^2 a^2) = 2\pi R_m \int_{-1}^{1} p_1(1, \cos\theta)\cos\theta \, d(\cos\theta). \tag{39}$$

From the orthogonality of the Legendre polynomials, only the coefficient of P_1 in (36) will contribute to the drag. Therefore Γ will not enter into this drag, and the pressure drag on the sphere is insensitive to the conductivity of the sphere to $O(R_m)$. Substitution of (36) into (39) gives

$$D_p = A_2 R_m(4\pi/5), \tag{40}$$

or $$\mathbf{F}_p = -\tfrac{2}{5}\pi B_0^2 \sigma a^3 \mathbf{U}_0 e^{i\omega t}, \tag{41}$$

and this is the instantaneous mechanical force on the sphere.

To obtain the drag due to electromagnetic stresses, the Maxwell-stress tensor must be integrated over the surface of the sphere. Since the sphere normal is radial, and the force in the z-direction is sought, this becomes

$$F_n = -2\pi a^2 \mu H_0^2 R_m \int_{-1}^{1} \mathbf{h}_1 \cdot \hat{\mathbf{r}} \, d(\cos\theta) = -(\quad)\int_{-1}^{1}(1-3\Gamma)P_2 \, d\cos\theta = 0. \tag{42}$$

Thus the magnetic stress tensor makes no contribution to the force on the sphere to $O(R_m)$.

At this point it is worth while to comment on the work of Ludford & Murray (1960), who treated the uniform motion of a *magnetized* sphere. Their results differ from those obtained here in three respects: the drag is not independent of the sphere conductivity to $O(R_m)$, the Maxwell-stress drag does not vanish to $O(R_m)$, and they state that an expansion in R_m may be non-uniform. These results are correct only for the case studied in that paper, namely when the sphere is the source of the magnetic field. When the sphere is magnetized, then the induced current and hence the drag depend upon the sphere conductivity through the electromagnetic boundary conditions at the surface of the sphere. This follows from the fact that their applied field (i.e. the magnetic field of the sphere) must follow the field in the fluid instantly, due to the continuity of **B**. However, in the present case, where the field is applied from infinity, the disturbance field in the fluid attenuates before it can reach the source of the applied field. It is precisely this different response of the applied field to the induced field that produces a contribution from the magnetic-stress tensor. Their series in R_m may be non-uniform while the present one is not for the same reason: their source field must follow the fluid field, and thus the $\mathbf{j} \times \mathbf{B}$ term must involve the perturbed fluid field in both $\mathbf{j}$ and $\mathbf{B}$ to first order; this is not so in the case under consideration, where only $\mathbf{j}$ is involved.

Now that the drag on the sphere is known, the energy balance of the system will be checked; this will yield another method of computing the drag. Energy is being dissipated because of the mechanical work that must be done to keep the sphere oscillating:

$$\partial W_{\text{mech}}/\partial t = \mathbf{F}.\mathbf{U}_0 = -\tfrac{2}{5}\pi B_0^2 \sigma a^3 U_0^2. \tag{43}$$

In the fluid, there is a motional e.m.f. which acts as a battery. The electric field which generates it $\mathbf{E}_{\text{mot}}$ is doing work at a rate

$$\frac{\partial W}{\partial t} = \int_V \mathbf{J}.\mathbf{E}_{\text{mot}}\, d\tau. \tag{44}$$

The current can be written as

$$\mathbf{J} = \sigma(\mathbf{E}+\mathbf{E}_{\text{mot}}), \tag{45}$$

where $\nabla \times \mathbf{E} = -\mu\, \partial H/\partial t$, so that

$$\int \mathbf{J}.\mathbf{E}_{\text{mot}}\, d\tau = \int \frac{J^2}{\sigma}\, d\tau - \int \mathbf{J}.\mathbf{E}\, d\tau. \tag{46}$$

That is, the work done consists of two parts: ohmic heating, and energy fed into the electromagnetic field. Using Maxwell's equations and a vector identity, the right-hand side of (46) can be rewritten as

$$-\int_V \mathbf{J}.\mathbf{E}\, d\tau = \frac{1}{2}\frac{\partial}{\partial t}\int_V (\mu H^2 + kE^2)\, d\tau + \int_{S_V} (\mathbf{E}\times\mathbf{H}).d\mathbf{S}. \tag{47}$$

The two terms on the right represent the rate of increase of electromagnetic field energy, and the rate of flow of this energy across the bounding surface. Therefore, since the e.m.f. is being driven by the mechanical force put into the system to move the sphere,

$$\mathbf{F}.\mathbf{U}_0 = \int \frac{J^2}{\sigma}\, d\tau + \frac{1}{2}\frac{\partial}{\partial t}\int (\mu H^2 + kE^2)\, d\tau + \int (\mathbf{E}\times\mathbf{H}).d\mathbf{S}. \tag{48}$$

In evaluating (64), we keep terms of $O(R_m)$ only and obtain

$$\mathbf{F}.\mathbf{U}_0 = \frac{R_m^2 H_0^2}{\sigma a^2}\int \mathbf{j}_1^2\, d\tau + \mu H_0^2 R_m \int (\mathbf{h}_1.\hat{\mathbf{z}})\, d\tau + R_m B_0 U_0 H_0 \int (\mathbf{e}_1 \times \hat{\mathbf{z}}).d\mathbf{S}. \tag{49}$$

The first integral is not dropped as being of $O(R_m^2)$ because an expansion in the magnetic Reynolds number is essentially an expansion in the conductivity, and thus (R_m^2/σ) is of $O(R_m)$ in that it approaches zero as σ, not as σ^2. The last two integrals vanish, again due to the orthogonality of the Legendre polynomials. Thus (49) reduces to

$$\int \frac{J^2}{\sigma}\, d\tau = -\tfrac{2}{5}\pi B_0^2 \sigma a^3 U_0^2 = \mathbf{F}\,.\,\mathbf{U}_0. \tag{50}$$

We see, just as Reitz & Foldy (1961) found for the case of uniform motion, the MHD drag on the sphere to $O(R_m)$ is equal to the Joule loss in the fluid divided by the velocity of the sphere, which loss may be computed directly from the applied magnetic field, and the zero-order, inviscid velocity.

6. Strong magnetic fields

If first-order quantities are adequate to describe the motion and fields in the fluid, it must be shown that a power series expansion is physically meaningful. That is, the induced velocity $\mathbf{V}_1$ and the induced magnetic field $\mathbf{H}_1$ must be at least an order of magnitude smaller than the fields present when the fluid is non-conducting

$$H_1/H_0 \sim R_m \ll 1, \quad V_1/V_0 \sim \alpha\beta R_m \ll 1. \tag{51}$$

Note that our condition for the smallness of the induced velocity is more easily satisfied than is the similar condition for uniform motion: $V_1/V_0 \sim \beta R_m \ll 1$ (Reitz & Foldy 1961).

For a 0·01 m sphere in mercury ($\sigma = 10^6$ mhos/m) oscillating with an amplitude of 10^{-3} m at 100 rad/sec, $R_m = 10^{-3}$ so that the induced magnetic field will always be small compared with the applied field, a result that is independent of the strength of the applied field. Substituting these numbers, the condition on the velocity becomes

$$\alpha\beta R_m = \frac{B_0^2 \sigma}{\rho\omega} \approx 0{\cdot}6 B_0^2 \ll 1, \tag{52}$$

where the applied field B_0 is measured in Wb/m². Then the present results are valid for fields of less than about 0·4 Wb/m² in mercury.

Thus for large fields we no longer have small perturbations of the fluid dynamic solution. Physically, what is happening is that the Lorentz force

$$\sigma(\mathbf{V}\times\mathbf{B}_0)\times\mathbf{B}_0 = \pm\,\sigma V B_0^2\,\hat{\boldsymbol{\rho}}$$

($\hat{\boldsymbol{\rho}}$ is the unit vector normal to the z-axis) opposes the flow of fluid around the sphere in the direction normal to $\mathbf{B}_0$ and the fluid transmits this body force to the sphere, where it appears as MHD pressure drag. In so acting, this force reduces the value of V_ρ, and thus of the induced current, i.e.

$$(\mathbf{V}_0 + \mathbf{V}_1)\times\mathbf{B}_0 < \mathbf{V}_0\times\mathbf{B}_0. \tag{53}$$

Since the actual current will be less than that computed by $\mathbf{J}_1 = \sigma\mathbf{V}_0\times\mathbf{B}_0$, the drag will also be smaller. From (51) it is seen that the correction is of order $\alpha\beta R_m$. This correction by itself, however, will still not give a correct value for the drag, since by using $(\mathbf{V}_0+\mathbf{V}_1)$ the current would be underestimated. In fact, if the drag is corrected just to $O(\alpha\beta R_m)$ it will vanish for some value of B_0^2, which is physically impossible. A second drag correction must be computed; this will be of $O(\alpha\beta R_m)^2$. Thus to $O(R_m)$ the induced velocity, and hence the current and the drag, must be calculated as an alternating power series in $\alpha\beta R_m$. In particular, the drag may be written as

$$\mathbf{F} = -\tfrac{2}{5}\pi B_0^2 \sigma a^3 \mathbf{U}_0[1+\sum_n(-1)^n f^{(n)}]\, e^{i\omega t}. \tag{54}$$

To evaluate $f^{(n)}$, it must first be shown that the magnetic stress makes no contribution to the drag to order $R_m(\alpha\beta R_m)^n$. Assuming the sphere to have the same permeability as the fluid, then at the surface of the sphere the normal component of $\mathbf{H}$ is continuous, provided it can be derived from a potential inside the sphere. Since the sphere is a dielectric, no true currents can exist inside it, and H satisfies this condition. Since the stress integral reduces to $O(R_m)$ to the integral of the normal component of $\mathbf{H}_1$ over the sphere (42), then, if this component is continuous, there can be no net force on the sphere due to magnetic stresses across its surface.

Thus all the MHD drag appears as pressure drag. Ludford & Murray (1960) have shown that the pressure drag due to the Lorentz force may be computed from

$$F = \pi\sigma B_0^2 a^3 U_0 \int_a^\infty \int_0^\pi (2f_r \cos\theta + f_\theta \sin\theta)\sin\theta\, d\theta\, \frac{dr}{r}, \tag{55}$$

where $\mathbf{f}$ is the dimensionless Lorentz force. For our case,

$$\mathbf{f}^{(n)} = (\mathbf{v}^{(n)} \times \mathbf{b}_0) \times \mathbf{b}_0, \tag{56}$$

where the correction to the velocity is computed from

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}^{(n)}) = \alpha\beta R_m \frac{\partial \mathbf{j}^{(n-1)}}{\partial z}, \tag{57}$$

and the correction to the current is computed from

$$\mathbf{j}^{(n)} = \mathbf{v}^{(n)} \times \mathbf{b}_0. \tag{58}$$

Note that for $n \geqslant 1$, $\nabla^2 \mathbf{j}^{(n)} \neq 0$, because the current is being generated by a non-potential velocity. Proceeding in this manner we obtain

$$\mathbf{J}_1^{(1)} = \mathbf{J}_1\{i\alpha\beta R_m(\tfrac{5}{2}\cos^2\theta - \tfrac{7}{5})\}, \tag{59}$$

$$\mathbf{f}^{(1)} = -i\alpha\beta R_m(1{\cdot}08), \tag{60}$$

$$\mathbf{f}^{(2)} = (\alpha\beta R_m)^2(1{\cdot}15). \tag{61}$$

7. Description and analysis of experiment

The experimental arrangement is shown in figure 1. The small moving sphere is encased in a large dielectric spherical shell, which is filled with fluid. The moving sphere is suspended vertically between two springs, the lower of which is fixed at the far end. The upper spring is driven harmonically by a low-frequency loudspeaker. The equation describing the resultant motion of the sphere is

$$F_0 e^{i\omega t} = -m\omega^2 z_0 e^{i\omega t} + i\omega A z_0 e^{i\omega t} + (k_1+k_2) z_0 e^{i\omega t}, \tag{62}$$

where $F_0(t)$ is the input force, k_1 and k_2 are the upper and lower spring constants, z_0 is the amplitude of oscillation of the sphere, and A, the coefficient of dissipation (drag) has the dimensions of (force/velocity). Taking the real part of (62), and writing the input force as the product of k_1 and the input amplitude b_0

$$b_0 k_1 \cos \omega t = [(k_1 + k_2) - m\omega^2] z_0 \cos \omega t - \omega A z_0 \sin \omega t. \tag{63}$$

At resonance,
$$\omega = \omega_0 = \sqrt{\{(k_1 + k_2)/m\}}. \tag{64}$$

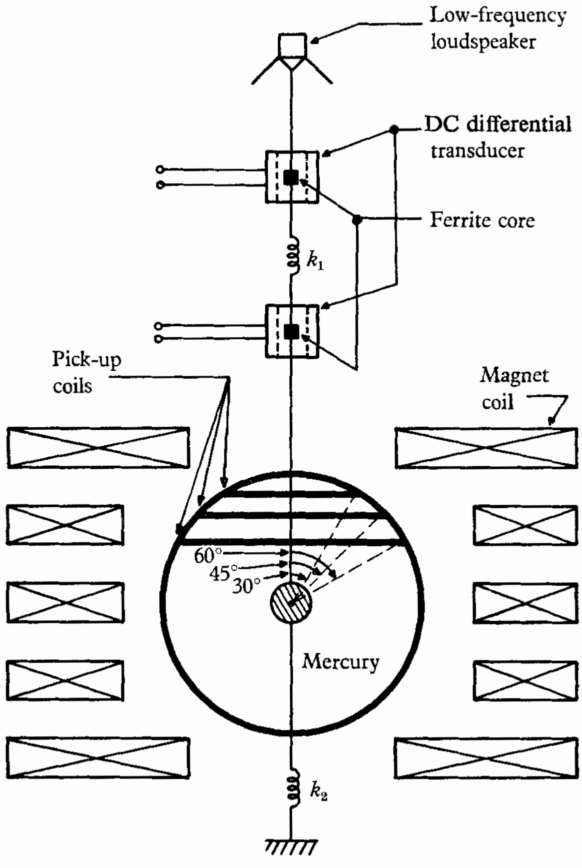

FIGURE 1. Experimental assembly.

Substitution of (64) into (63) shows that at resonance the drag coefficient is linearly proportional to the ratio of input to sphere amplitude

$$A = b_0 k_1 / z_0 \omega_0. \tag{65}$$

The input and sphere amplitudes were measured with Hewlett–Packard DC differential transducers, which generate a DC voltage proportional to linear displacement. For this system, there are sinusoidal outputs at resonance, and the input and sphere amplitudes are 90° out of phase. This fact was used as a sensitive test to determine when the system was at resonance: the output of one DCDT was placed on the horizontal deflexion plates of an oscilloscope, and the output of the other on the vertical plates. When the resultant Lissajous ellipse was vertical, the system was at resonance.

The measured drag consists of three parts: the tare drag A_t, the viscous drag A_η and the MHD drag A_M. The tare drag was mainly due to sliding friction at the point where the wire left the case. The experimental apparatus was calibrated by measuring the viscous drag when the fluid was water. The system was first run at resonance with no fluid; this determined A_t. The case was then filled with water, the new resonant frequency measured, and the new drag determined. The difference between this drag, (A_t+A_η), and the air drag A_t should be the Stokes drag

$$A_\eta = 6\pi a\eta(1+a/\delta), \tag{66}$$

where δ, the thickness of the viscous boundary layer, is the e-folding distance of the vorticity, and is given by

$$\delta = \sqrt{(2\eta/\rho\omega)} = a\sqrt{(2\alpha/R_e)}. \tag{67}$$

As a further check on the accuracy of our system, the induced mass was also measured. This added inertial mass arises from the fact that the motion of the sphere forces the water to oscillate; thus all of the input force does not go into accelerating the sphere

$$F_{\text{acc}} = m\ddot{z}_0 = (m_s+\delta m)\ddot{z}_0.$$

This induced mass was determined by the difference between the resonant frequencies for air and for water

$$\delta m = \left.\frac{k_1+k_2}{\omega_0^2}\right)_{\text{water}} - \left.\frac{k_1+k_2}{\omega_0^2}\right)_{\text{air}}, \tag{68}$$

and is to be compared with the Stokes value

$$\delta m = \tfrac{4}{3}\pi\rho a^3(\tfrac{1}{2}+\tfrac{9}{4}\delta/a). \tag{69}$$

When the MHD experiment with mercury was performed, it was essential to determine if, in fact, the inviscid potential velocity may be used to compute the current. This was done by measuring the e.m.f. induced in pick-up coils placed at polar angles of 30°, 45°, 60° in the azimuthal plane. If it is correct to use the potential velocity, the current is given by (26), and the associated time-dependent magnetic field will generate an azimuthal electric field given by (34). Since the angular dependence of this electric field is $\sin\theta\cos\theta$, then the angular dependence of the e.m.f. induced in a coil coplanar with $\mathbf{e}$ and concentric with the line of motion is

$$\mathscr{E}(\theta) = \sin^2\theta\cos\theta. \tag{70}$$

100-turn pick-up coils were used, and were loaded to make their resistances equal. The coil signals were increased with a power amplifier, and measured with a lock-in amplifier. Due to the fact that the lock-in amplifier contained an internal phase shifter, it was possible to measure the e.m.f. associated with $\mathbf{J}_1 = \sigma\mathbf{V}_0\times\mathbf{B}_0$, rather than the current that is actually present

$$\mathbf{J} = \mathbf{J}_1(1+i\alpha\beta R_m j^{(1)}).$$

8. Experimental results and analysis

The experiment was performed with a lucite ball in a spherical lucite case with walls 0·25 in. thick. The inner radius of the case was 10 times the radius of the ball. This effectively removes wall effects from the fluid dynamic motion because it can be shown (Lamb 1952) that for this problem the effect of the bounding wall on the fluid velocity, viscous drag and added mass varies as the cube of the ratio of the two radii.

The driver of the system was a woofer produced by Acoustical Research Corporation (model AR-3), which has a nominal throw of 0·250 in., and bottoms out at 0·50 in. It was vibrated by an audio oscillator, and has less than 1 % harmonic distortion at the current input used. The frequency of oscillation was measured with a stroboscope, and at the frequency used (typically 20–30 c/sec) this meter has an error of $\pm 0{\cdot}083$ c/sec.

The magnetic field was generated by five Magnion Plasmaflux coils, supplied by a rectifier having less than 1 % peak-to-peak ripple. The magnetic field was designed to have less than 1 % variation in the longitudinal field, and to keep the radial field less than 1 % of the longitudinal field, within a sphere of radius 4 in. During the experiment, the magnetic field was measured with a rotating coil gaussmeter, which has an accuracy of 1 %.

The values for the viscosity and density of water were taken from the *Handbook of Chemistry and Physics*, as were the density and conductivity of mercury. The value for the conductivity was checked by measuring the resistance of a column of mercury of known length and cross-section; the results agreed with the book value within experimental error. The viscosity of mercury at the temperature of the room was calculated from the work of Andrade (1934).

The experiment was performed using a lucite ball of radius $0{\cdot}9575 \pm 0{\cdot}0064$ cm and sphericity 0·005. The results in the absence of a magnetic field are contained in table 1.

	Water	Mercury
Air resonant frequency	$27{\cdot}58 \pm 0{\cdot}083$ c/sec	$31{\cdot}00 \pm 0{\cdot}083$ c/sec
Fluid resonant frequency	$25{\cdot}33 \pm 0{\cdot}083$ c/sec	$20{\cdot}00 \pm 0{\cdot}083$ c/sec
Boundary layer, δ/a	$1{\cdot}13 \times 10^{-2}$	$3{\cdot}43 \times 10^{-3}$
Air resonant mass	$10{\cdot}49 \pm 0{\cdot}21$ g	$17{\cdot}58 \pm 0{\cdot}34$ g
Fluid resonant mass	$12{\cdot}44 \pm 0{\cdot}24$ g	$42{\cdot}24 \pm 0{\cdot}81$ g
Tare drag, A_t	$(96{\cdot}9 \pm 1{\cdot}72) \times 10^3$ N/m/sec	$(210{\cdot}0 \pm 3{\cdot}65) \times 10^3$/N/msec
Fluid drag, $A_t + A_\eta$	$(102{\cdot}4 \pm 1{\cdot}77) \times 10^3$ N/m/sec	$(273{\cdot}9 \pm 4{\cdot}03) \times 10^3$/N/msec
Theoretical induced mass	$1{\cdot}928 \pm 0{\cdot}023$ g	$24{\cdot}71 \pm 0{\cdot}30$ g
Experimental induced mass	$1{\cdot}95 \pm 0{\cdot}24$ g	$24{\cdot}66 \pm 0{\cdot}86$ g
Theoretical viscous drag	$(15{\cdot}45 \pm 0{\cdot}16) \times 10^3$ N/m/sec	$(61{\cdot}59 \pm 0{\cdot}64) \times 10^3$ N/m/sec
Experimental viscous drag	$(15{\cdot}5 \pm 2{\cdot}47) \times 10^3$ N/m/sec	$(63{\cdot}9 \pm 5{\cdot}44) \times 10^3$ N/m/sec

TABLE 1. Hydrodynamic results

Thus agreement is obtained within experimental error for the value of Stokes's drag and induced mass for a sphere executing small-amplitude oscillations. In fact, this is the first time this measurement has been performed for a sphere with amplitude conditions appropriate to compare with Stokes's analysis. These results are (after all these years) a quantitative verification of Stokes's solution for an oscillating sphere.

It may be thought that the relatively high percentage error in the experimental value of the viscous drag should be reduced. This error arises because the viscous drag is small compared with the tare drag. A small percentage error in the total drag becomes a large percentage error when the two nearly equal drags are subtracted to obtain the viscous drag. The error could be reduced by increasing the size of the moving sphere, thus increasing the viscous drag. This would indeed have been done if this were an experiment designed solely to test the equations of viscous oscillatory motion. However, we are interested principally in measuring the MHD drag. Since this drag will be determined by subtracting the drag without magnetic field ($A_t + A_\eta$) from the drag with a magnetic field ($A_M + A_t + A_\eta$), it is desirable to keep this 'background' drag as small as possible. The main contribution to the tare drag comes from the frictional force on the wire where it leaves the case. Since this cannot be reduced without permitting the fluid to leak out during the experiment, the viscous drag must be kept small.

It is important to note that agreement was found with Stokes's theory despite the existence of a small secondary flow. This is a streaming effect (non-oscillatory) which drives fluid away from the sphere along the axis of oscillation. The fluid flows out until it reaches the case, then flows down the wall, and returns to the sphere in the equatorial plane. The flow was made visible by dropping ink into the water, and was observed to commence outside the viscous boundary layer at a distance of approximately δ/α, as predicted (Andres & Ingard 1953; Stuart 1963). The drag agreement shows that the contribution of the secondary flow to the drag is unimportant compared with the oscillatory flow. Although this is known theoretically and experimentally for certain distant boundaries, one could not be sure of this *a priori* for the given boundary conditions.

When the magnetic field was turned on, and the sphere was filled with mercury, the validity of using the potential velocity to compute the current was checked by measuring the e.m.f.'s picked up in the three coils. The data are presented in table 2; the errors represent standard deviations. The measured angular dependence agrees with that predicted by (70), so that the use of the inviscid velocity to compute the current, and hence the drag, is shown to be correct.

	$\mathscr{E}(60°)/\mathscr{E}(45°)$	$\mathscr{E}(45°)/\mathscr{E}(30°)$	$\mathscr{E}(60°)/\mathscr{E}(45°)$
Theoretical	$1{\cdot}060 \pm 0{\cdot}011$	$1{\cdot}632 \pm 0{\cdot}016$	$1{\cdot}732 \pm 0{\cdot}017$
Experimental	$1{\cdot}085 \pm 0{\cdot}056$	$1{\cdot}626 \pm 0{\cdot}084$	$1{\cdot}755 \pm 0{\cdot}090$

TABLE 2. Ratio of coil e.m.f.'s at different polar angles

When examining oscilloscope pictures of the induced e.m.f., smaller waves of higher amplitude were observed to modulate the 20 c/sec signal created by $\mathbf{h}_1$. These represent the e.m.f. induced by damped Alfvén waves, and will be discussed in a future paper. It was the presence of these waves that necessitated the use of the lock-in amplifier, with the relatively large error of 5 %, to measure the induced e.m.f.

The experimental MHD drag values are plotted against B_0^2 in figure 2. Very good agreement is found by the theory which includes terms of $O(\alpha\beta R_m)^2$. In the graph, the theoretical curve is obtained from

$$A_M = \tfrac{2}{5}\pi B_0^2 \sigma a^3\{1 - 1\cdot08\alpha\beta R_m + 1\cdot15(\alpha\beta R_m)^2\}, \tag{71}$$

which, for the given experimental apparatus becomes

$$A_M = 1135B_0^2(1 - 0\cdot651B_0^2 + 0\cdot388B_0^4) \times 10^3\,\text{N/m/sec}, \tag{72}$$

where B_0 is in units of Wb/m^2. The error bars in the graph are derived from the percentage error. Note that the percentage error in the experimental value of A_M decreases as A_M increases and becomes larger than A_l.

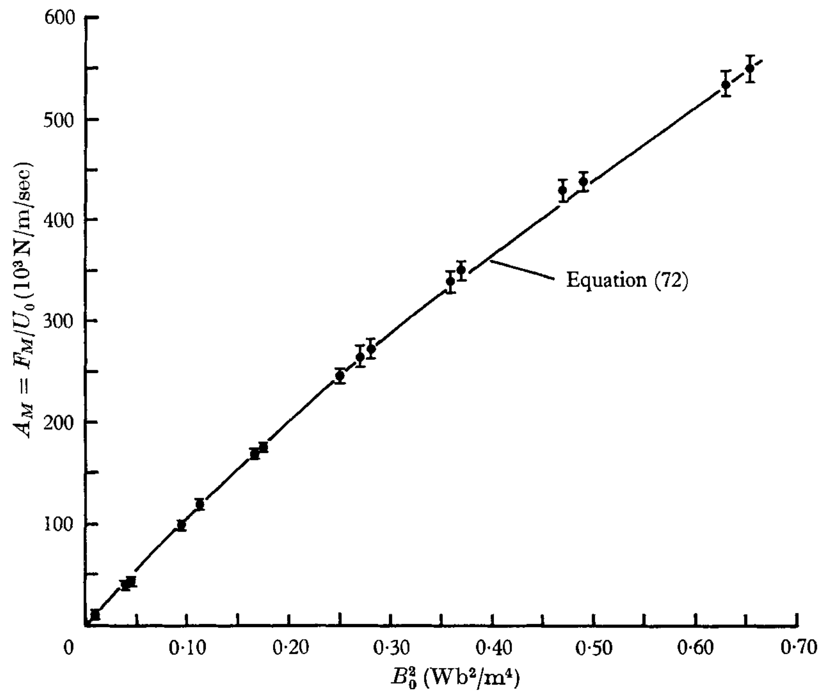

FIGURE 2. Variation of magnetic drag with magnetic field.

9. Conclusion

Analytical solutions have been obtained for the MHD problem of a dielectric sphere executing small-amplitude oscillations in mercury with an aligned magnetic field, under the assumption of low magnetic Reynolds number. The angular dependence of the induced electric field was measured, and the results justified the use of the inviscid velocity to compute the induced current. The magnetic drag was measured, and showed excellent agreement with the theoretical value, which was extended to include the régime where the induced velocity was not small compared with the inviscid velocity.

In the calibration of the experimental apparatus, the induced mass and viscous drag on an oscillating sphere were measured for the first time within the limits of the Stokes small-amplitude approximation, and agree with Stokes's theoretical values.

To the author's knowledge, this is the first quantitative verification of a perturbation solution to the linearized MHD equations.

This paper is taken from a thesis submitted as partial fulfilment of the requirements for the degree of Doctor of Philosophy in the Faculty of Pure Science, Columbia University. I would like to thank Professor Robert A. Gross, my thesis advisor, for his advice and encouragement during the course of this work. I would also like to thank Maurice Cea and John Osarczuk for their help in constructing the experimental apparatus and for their assistance during the experimental runs. This study was supported by the National Science Foundation under grant NSF GP-554.

REFERENCES

ANDRADE, E. N. DA C. 1934 *Phil. Mag.* **17**, 698.

ANDRES, J. M. & INGARD, U. 1953 *J. Acoust. Soc. Amer.* **25**, 928.

CARSTENS, M. R. 1952 *Trans. Amer. Geophys. Un.* **33**, 713.

CHESTER, W. 1952 *J. Fluid Mech.* **3**, 304.

CHILDRESS, W. S. 1961 *California Institute of Technology, Jet Propulsion Lab. Tech. Rep.* no. 32–200.

DORMAN, L. I. & MIKHAILOV, Y. M. 1963 *Sov. Phys. JETP*, **16**, 531.

GREEN, G. 1833 *Edin. Trans.*, p. 54.

LAMB, H. 1952 *Hydrodynamics*, 6th edn. New York: Dover.

LUDFORD, G. S. S. & MURRAY, J. D. 1960 *J. Fluid Mech.* **7**, 516.

MAXWORTHY, T. 1962 *Proc. Heat Transfer and Fluid Mechanics Inst.*, p. 197. Stanford University Press.

MORSE, P. M. & FESHBACH, H. 1953 *Methods of Theoretical Physics.* New York: McGraw-Hill.

MOTZ, R. O. 1964 *Columbia University Plasma Lab. Rep.* no. 10.

REITZ, J. R. & FOLDY, L. L. 1961 *J. Fluid Mech.* **11**, 133.

SINGH, M. P. 1965 *Phys. Fluids*, **8**, 797.

STOKES, G. G. 1843 *Camb. Trans.* **8**, 105.

STOKES, G. G. 1850 *Camb. Trans.* **9**, 8.

STUART, J. T. 1963 *Laminar Boundary Layers* (Ed. L. Rosenhead), ch. 7. Oxford University Press.

STUART, J. T. & WOODGATE, L. 1955 *Phil. Mag.* **46**, 40.